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Sets of Integers with Missing Differences

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This paper deals with the problem of finding the maximal density $\mu(M)$ of sets of integers in which the differences given by a set M do not occur (M-sets). Some general estimates are given, $\mu(M)$ is compared to other set functions, and expressions for $\mu(M)$ are given for most members of the families $\{1, j, k\}$ and $\{1, 2, j, k\}$.

1. INTRODUCTION

With the exception of minor modifications, we shall use the notation adopted by Cantor and Gordon in [1]. More specifically, if S is a set of nonnegative integers, we define the upper and lower densities of S, denoted by $\bar{\delta}(S)$ and $\underline{\delta}(S)$, respectively, by

$$\bar{\delta}(S) = \overline{\lim_{n\to\infty}}\, S(n)/n, \qquad \underline{\delta}(S) = \underline{\lim}_{n\to\infty}\, S(n)/n,$$

where $S(n)$ is the number of elements in $\{0, 1,..., n\} \cap S$, a notation that will be used throughout this paper. We say that S has density $\delta(S)$ when $\underline{\delta}(S) = \bar{\delta}(S) = \delta(S)$.

If M is a given set of positive integers, S will be called an *M-set* if $a, b \in S$ implies $a - b \notin S$, and sup $\bar{\delta}(S)$, taken over all M-sets S, will be denoted by $\mu(M)$. The problem treated in this paper was posed by Professor Motzkin [2], and is that of determining the quantity $\mu(M)$.

The characteristic sequence of an M-set S, written as a binary string, will be called an *M-sequence* in this paper, and we shall use the same symbols for M-sets and M-sequences, in view of the natural correspondence. Relating to binary sequences and strings, we shall be using juxtaposition to denote concatenation and exponentiation to denote repeated concatenation.

If A is a set of numbers, we shall use $|A|$ for the cardinality of A and $A \pm x$ for the set $\{a \pm x: a \in A\}$.

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Finally, if x is a real number, we shall use the following notation:

$[x]$ = largest integer not exceeding x;

$\{x\} = x - [x]$, the fractional part of x;

$\| x \| = \min(\{x\}, 1 - \{x\})$, the distance of x to the nearest integer.

2. Some General Results

It has been shown [1, Theorem 1] that

$$\mu(M) \geqslant \sup_{(k,m)=1} (1/m) \min_i | km_i |_m ,$$

where m_i are the elements of M and $| x |_m$ denotes the absolute value of the absolutely least remainder of $x \pmod m$. The following remark, mentioned here without proof, gives other equivalent expressions for the right-hand side of this inequality.

Remark 1. Let $M = \{m_1, m_2, ..., m_n\}$ and

$$d_1(M) = \sup_{x \in (0,1)} \min_i \| xm_i \|,$$

$$d_2(M) = \sup_{(k,m)=1} (1/m) \min_i | km_i |_m ,$$

$$d_3(M) = \max_{\substack{m = m_j + m_l \\ 1 \leqslant k \leqslant m/2}} (1/m) \min_i | km_i |_m .$$

Then $d_1(M) = d_2(M) = d_3(M)$ and we denote this common value by $d(M)$.

Since $\mu(M) \geqslant d(M)$ and, in fact, equality holds when $| M | = 2$ as shown in [1], the question has arisen whether equality always holds and whether the denominator of $\mu(M)$ divides the sum of two elements of M, as is the case with $d(M)$. In Sections 3 and 4 we shall see that equality holds for all the known families with $| M | = 3$, but fails for some infinite families with $| M | = 4$, where in fact the denominator of $\mu(M)$ does not divide the sum of two elements of M. (Note: By [1, Theorem 5], $\mu(M)$ is always a rational number.)

The following theorem establishes two simple lowerbounds for $\mu(M)$.

THEOREM 1. *If* $M = \{m_1, m_2, ..., m_n\}$ *with* $m_i < m_{i+1}$, *then*

(a) $\mu(M) \geqslant 1/(n + 1)$, *and*

(b) $\mu(M) \geqslant m_1/(m_1 + m_n)$.

Proof. (a) An M-sequence of density at least $1/(n+1)$ can be constructed by the following procedure:

Step 1. Starting with a sequence of blanks, numbered 0, 1, 2,..., we place a 1 in position 0 and 0's in all positions m_i.

Step j. We go to the next blank position k, place a 1 in that position and 0's in all positions $k + m_i$.

By repeating step j ($j = 2, 3, \ldots$) we obtain an M-sequence in which each 1 corresponds to at most n 0's, that is, of density at least $1/(n+1)$. (Note: If $M = \{1, 2, \ldots, n\}$ then in fact $\mu(M) = 1/(n+1)$.)

(b) Let S be a periodic binary sequence with period $1^{m_1}0^{m_n}$, period length $m_1 + m_n$. It is clear that S is an M-sequence of density $m_1/(m_1 + m_n)$. ∎

The next theorem gives a lowerbound for $d(M)$, of interest primarily for its limit as $m_3 \to \infty$.

THEOREM 2. *If* $M = \{m_1, m_2, m_3\}$, $m_1 < m_2 < m_3$, $g = \mathrm{GCD}(m_1, m_2)$, *then* $d(M) \geqslant 1/[2 + (2m_2/m_3)]$ *if* m_1/g, m_2/g *both odd, and* $d(M) \geqslant [(m_1 + m_2 - g)/2]/[m_1 + m_2 + (2m_1m_2/m_3)]$ *if* m_1/g, m_2/g *of opposite parity.*

Proof. The first expression for $d(M)$ in Remark 1 is equivalent to

$$d(M) = \sup\{\alpha \in (0, \tfrac{1}{2}): \text{there exists } x \in [0, 1] \text{ such that } \| xm_i \| \geqslant \alpha, i = 1, 2, 3\}$$

or, if we let $A_i(\alpha) = \{x \in [0, 1]: \| xm_i \| \geqslant \alpha\}$,

$$d(M) = \sup\left\{\alpha \in (0, \tfrac{1}{2}): \bigcap_i A_i(\alpha) \neq \varnothing\right\}.$$

The proof is based on the observation that each set $A_i(\alpha)$ is the union of m_i disjoint intervals with centers $(2k-1)/2m_i$ ($k = 1, 2, \ldots, m_i$) and width $2\epsilon/m_i$, where $\epsilon = \frac{1}{2} - \alpha$. We then distinguish the two cases:

(i) If m_1/g, m_2/g are both odd, then for $i = 1, 2$ the $\{[(m_i/g) + 1]/2\}$th interval of $A_i(\alpha)$ has center $1/(2g)$, so that $A_1(\alpha) \cap A_2(\alpha) \neq \varnothing$ always. Consecutive intervals of A_3 have centers $1/m_3$ apart, so $\bigcap_i A_i(\alpha) \neq \varnothing$ provided that $\epsilon/m_2 + \epsilon/m_3 \geqslant 1/(2m_3)$, or, equivalently,

$$\alpha \leqslant 1/[2 + (2m_2/m_3)].$$

(ii) If m_1/g, m_2/g are of opposite parity, then the shortest distance between interval centers of $A_1(\alpha)$ and $A_2(\alpha)$ is $g/(2m_1m_2)$, and the respective intervals overlap by $\epsilon[(1/m_1) + (1/m_2)] - g/(2m_1m_2)$. This overlap intersects

$A_3(\alpha)$, which consists of intervals of width $2\epsilon/m_3$ and centers $1/m_3$ apart provided that

$$\epsilon\left(\frac{1}{m_1}+\frac{1}{m_2}\right)-\frac{g}{2m_1m_2}\geqslant\frac{1}{m_3}-\frac{2\epsilon}{m_3},$$

which reduces to

$$\alpha\leqslant\frac{(m_1+m_2-g)/2}{m_1+m_2+(2m_1m_2/m_3)}. \quad \blacksquare$$

From this theorem and in view of the expression for $\mu(\{m_1, m_2\})$ [1, Theorem 4], it follows that $d(\{m_1, m_2, m_3\})$ and consequently $\mu(\{m_1, m_2, m_3\})$ tend to $\mu(\{m_1, m_2\})$ as m_3 tends to infinity. This observation is generalized in the following result.

Remark 2. If $M=\{m_1, m_2, \ldots, m_{n-1}\}$ and $M'=\{m_1, m_2, \ldots, m_{n-1}, m_n\}$ $(m_i<m_{i+1})$ then $\mu(M')\to\mu(M)$ *as* $m_n\to\infty$.

Proof. Let S be a peiodic M-sequence of period length q and density $\delta(S)=\mu(M)=p/q$. Also let S' be a periodic sequence with period S^a0^b, where $a=[m_n/q]$ and $b=m_n-qa+m_{n-1}$. It can be verified that S' is an M'-sequence, which implies

$$p/q=\mu(M)\geqslant\mu(M')\geqslant\delta(S')=p[m_n/q]/(m_n+m_{n-1}).$$

Passing to the limit as $m_n\to\infty$ we obtain the stated result. $\blacksquare$

Unfortunately this result cannot be generalized to the case where more than one element of M tends to infinity; letting $M=\{i, j, i+j\}$, it can be easily verified that $\mu(M)\leqslant\frac{1}{3}$, hence $\mu(M)$ cannot approach $\mu(\{i\})=\frac{1}{2}$ as $j\to\infty$.

We conclude this section with an upperbound for $\mu(M)$, a result that will be referred to repeatedly in the following sections.

LEMMA 1. *Let M be a given set of positive integers, α a real number in the interval* $[0, 1]$, *and suppose that for any M-set S with $0\in S$ there exists a positive integer k (possibly dependent on S) such that $S(k)\leqslant(k+1)\alpha$. Then $\mu(M)\leqslant\alpha$.*

Proof. If T is any M-set, let $S_1=T-n_1$; where $n_1=\inf T$. Then S_1 is an M-set with $0\in S_1$, so there exists a positive integer k_1 such that

$$T(n_1+k_1)=S_1(k_1)\leqslant(k_1+1)\alpha.$$

For $i\geqslant 2$ let $S_i=T-n_i$, where $n_i=\inf\{x\colon x\in T \text{ and } x>n_{i-1}+k_{i-1}\}$. Then S_i is an M-set with $0\in S_i$, so there exists k_i such that

$$T(n_i+k_i)-T(n_{i-1}+k_{i-1})=S_i(k_i)\leqslant(k_i+1)\alpha.$$

Adding these inequalities and letting $N_j = k_1 + \cdots + k_j + j$, whereby $n_j + k_j \geqslant N_j - 1$, we obtain $T(N_j - 1) \leqslant N_j\alpha$. It follows that $\delta(T) \leqslant \alpha$ and $\mu(M) \leqslant \alpha$. ∎

3. The Family $M = \{1, j, k\}$

We shall now determine $\mu(M)$ for most of the sets $M = \{1, j, k\}$. The first theorem treats the case in which j is even.

Theorem 3. *If $M = \{1, j, k\}$, where $1 < j < k$ and j is even, and if $k = n(j + 1) + \bar{k}$ $(0 \leqslant \bar{k} \leqslant j)$, then*

(a) *$\mu(M) = (j/2)/(j + 1)$ if $\bar{k} = 1$ or j;*

(b) *$\mu(M) = \{(nj/2) + [\bar{k}/2]\}/(k + 1)$ otherwise, provided that $n \geqslant (j - \bar{k} - 2)/2$ when $\bar{k}$ is even and $n \geqslant (2j - \bar{k} - 3)/2$ when $\bar{k}$ is odd.*

Proof. (a) If $\bar{k} = 1$ or j, then the $\{1, j\}$-set of period $j + 1$ and density $\mu(\{1, j\})$ given in [1, Theorem 4] is also an M-set, therefore

$$\mu(M) = \mu(\{1, j\}) = (j/2)/(j + 1).$$

(b) We first consider the case in which $j \geqslant 4$ and $\bar{k}$ is even. Let X, Y be the binary strings $(10)^{j/2}0$ and $(10)^{j/2-1}0$, respectively, and let $\alpha = n - (j - \bar{k} - 2)/2$ and $\beta = n - \alpha + 1$. If S is a periodic binary sequence with period $X^\alpha Y^\beta$, it can be verified that S is an M-sequence, that S has period length $k + 1$ and that

$$\delta(S) = d = [(nj/2) + (\bar{k}/2)]/(k + 1),$$

which implies $\mu(M) \geqslant d$.

To show the reverse inequality, let S be any M-set with $0 \in S$, and let $\bigcup_{i=0}^{n-1} A_i \cup B$ be the decomposition of $\{0, 1, \ldots, k\}$ into disjoint sets $A_i = \{0, 1, \ldots, j\} + (j + 1)i$ and $B = \{0, 1, \ldots, \bar{k}\} + (j + 1)n$. By [1, Theorem 4], $| A_i \cap S | \leqslant j/2$; also $|(B - \{k\}) \cap S| \leqslant \bar{k}/2$ and $\{k\} \cap S = \varnothing$. Therefore $S(k) \leqslant (nj/2) + (\bar{k}/2)$, and, by Lemma 1, $\mu(M) \leqslant d$.

In the case where $j \geqslant 4$ and $\bar{k}$ is odd, let X, Y be as before, and let $\alpha = n - (2j - \bar{k} - 3)/2$ and $\beta = n - \alpha + 2$. Again, the binary sequence S with period $X^\alpha Y^\beta$ is an M sequence and

$$\delta(S) = d = [nj/2 + (\bar{k} - 1)/2]/(k + 1),$$

which implies $\mu(M) \geqslant d$.

For the reverse, let S be any M-set with $0 \in S$ and suppose that $S(k) > nj/2 + (\bar{k} - 1)/2$. Decomposing $\{0, 1, \ldots, k\}$ as before, this implies that

$| A_i \cap S | = j/2$, $B \cap S = \{0, 2, 4,..., \bar{k} - 1\} + n(j + 1)$, and $S(k) = nj/2 + (\bar{k} - 1)/2 + 1$. Considering now $A_{n-1} \cap S$ we observe that one pair of consecutive elements of A_{n-1} must be outside S, that is,

$$A_{n-1} \cap S = \{0, 2, 4,..., e_1 - 2, e_1 + 1, e_1 + 3,..., j - 1\} + (n - 1)(j + 1),$$

where e_1 is even and at least $\bar{k} + 1$. Applying a similar argument successively to A_{n-2}, A_{n-3},..., A_0 we obtain

$$A_0 \cap S = \{0, 2, 4,..., e_n - 2, e_n + 1, e_n + 3,..., j - 1\},$$

where e_n is even and at least e_1. The structures obtained for $A_{n-1} \cap S$ and $A_0 \cap S$ imply that $k + e_1 - \bar{k} + i \notin S$ $(i = 0, 2, 4,..., j - 2)$ and $k + i \notin S$ $(i = 0, 2, 4,..., e_1 - 2)$, respectively, hence $S(k+e_1-1)-S(k) \leqslant (e_1-\bar{k}-1)/2$. From this it follows that $S(k + e_1 - 1) \leqslant (nj/2) + (e_1/2)$, and since it can be verified that under the hypotheses

$$\frac{(nj/2) + (e_1/2)}{k + e_1} \leqslant \frac{nj/2 + (\bar{k} - 1)/2}{k + 1} = d,$$

the inequality $\mu(M) \leqslant d$ follows from Lemma 1.

To complete the proof we now consider the case in which $j = 2$ and $k \equiv 0 \pmod 3$. By considering a sequence with period $(100)^{k/3}0$ we obtain

$$\mu(M) \geqslant (k/3)/(k + 1).$$

On the other hand, decomposing $\{0, 1,..., k\}$ into $k/3$ consecutive triples A_i and $\{k\}$, we observe that for any M-set S with $0 \in S$ we must have $k \notin S$ and $| A_i \cap S | \leqslant 1$. Lemma 1 then implies

$$\mu(M) \leqslant (k/3)/(k + 1). \quad \blacksquare$$

Before we proceed to the next theorem, which deals with the case in which j is odd, we prove the following.

LEMMA 2. *If j is odd, $e \geqslant j - 1$ and T is a $\{1, j\}$-set such that $\{0, 2,..., j - 3\} \subset T$ and $\{1, 3,..., j - 2\} + 2e \subset T$ then*

$$T(2e - 1) \leqslant e - (j - 1)/2.$$

Proof. We write $2e = n(j - 1) + 2\bar{e}$, where $0 \leqslant 2\bar{e} \leqslant j - 3$ and $n \geqslant 2$, and decompose $\{0, 1,..., 2e - 1\} = \bigcup_{i=0}^{n-1} A_i \cup B$, where $A_i = \{0, 1,..., j - 2\} + (j - 1)i$ and $B = \{2e - 2\bar{e},..., 2e - 1\}$. We also let $C = \{2e, 2e + 1,..., 2e + j - 2\}$. The proof is by induction on n: for $n = 2$, the hypotheses for T

imply that $A_1 \cap T$ contains no odd numbers, and no even numbers larger than $j + 2\bar{e} - 2$. Therefore $|A_1 \cap T| = \lambda \leqslant \bar{e}$, and so λ odd numbers of B are outside T. But $B \cap T$ has no even numbers, hence $|B \cap T| \leqslant \bar{e} - \lambda$. Since also $|A_0 \cap T| = (j-1)/2$ in view of the hypotheses, it follows that $T(2e-1) \leqslant e - [(j-1)/2]$. For $n > 2$, now, suppose $|A_i \cap T| = [(j-1)/2] - a_i$. If $a_i = 0$ for some i, then $A_i \cap T$ must consist of numbers of the same parity, in which case the desired result follows from the induction hypothesis. We can therefore assume that all a_i are positive. In particular, the set $A_1 \cap T$ must consist of $[(j-1)/2] - a_1$ even numbers, since it cannot include any odd ones. Similarly, the set $A_2 \cap T$ can contain at most a_1 odd numbers, so it includes at least $[(j-1)/2] - a_1 - a_2$ even ones and so on. Finally, the set $A_{n-1} \cap T$ has at least $[(j-1)/2] - a$ even numbers, where $a = \sum_{i=1}^{n-1} a_i$, so that $[(j-1)/2] - a$ odd numbers of B are outside T. Since the hypothesis on $C \cap T$ implies that $B \cap T$ has no even numbers, we conclude that $|B \cap T| \leqslant \bar{e} - [(j-1)/2] + a$. Therefore

$$T(2e-1) \leqslant \frac{j-1}{2} + \sum_{i=1}^{n-1} \left(\frac{j-1}{2} - a_i\right) + \bar{e} - \frac{j-1}{2} + a$$

$$= e - \frac{j-1}{2}. \quad \blacksquare$$

THEOREM 4. *If $M = \{1, j, k\}$, where $1 < j < k$ and j is odd, then*

(a) $\mu(M) = \frac{1}{2}$ *if k is odd,*

(b) $\mu(M) = (k/2)/(k+j)$ *otherwise, provided that* $k \geqslant j(j-1)/2$.

Proof. (a) If k is odd, then $\{0, 2, 4, \ldots\}$ is an M set of density $\frac{1}{2}$.

(b) If k is even, let S be a periodic binary sequence with period $(10)^{k/2}\, 0^j$. It is easily verified that S is an M sequence and that

$$\delta(S) = d = (k/2)/(k+j).$$

Therefore $\mu(M) \geqslant d$.

To show the reverse inequality, let S be an M set with $0 \in S$. If the string $(10)^{(j+1)/2}$ is not a substring of S, then

$$\delta(S) \leqslant [(j-1)/2]/(j+1) \leqslant d.$$

We can therefore assume, without loss of generality, that $\{0, 2, 4, \ldots, j-1\} \subset S$; in view of Lemma 1, it will be sufficient to show that $S(k+j-1) \leqslant k/2$. If we let $D = \{k, k+1, \ldots, k+j-1\}$, this assumption implies that $k, k+2, k+4, \ldots, k+j-1 \notin S$, that is, $D \cap S$ consists of odd numbers. If $|D \cap S| = (j-1)/2$, then Lemma 2 applied for $e = k/2$ implies that $S(k-1) \leqslant (k/2) - [(j-1)/2]$, hence $S(k+j-1) \leqslant k/2$. We can therefore restrict our attention to the case in which $|D \cap S| = m < (j-1)/2$.

We write $\{0, 1,..., k-1\} = \bigcup_0^m C_i$, where C_0 consists of the first $k - m(j-1)$ nonnegative integers and each successive C_i consists of the next $j-1$ integers. We note that $j - 1 \in C_0$, as a consequence of the restriction $k \geqslant j(j-1)/2$. Considering now C_m, we observe that since $D \cap S$ has m odd numbers, $C_m \cap S$ can have at most $[(j-1)/2] - m$ even numbers. If, on the other hand, $C_m \cap S$ has $(j-1)/2$ odd numbers, Lemma 2 applied for $e = (k/2) - [(j-1)/2]$ implies that $|(C_0 \cup C_1 \cup \cdots \cup C_{m-1}) \cap S| \leqslant k/2 - j + 1$, hence $S(k + j - 1) \leqslant k/2$. We can therefore assume that $|C_m \cap S| = [(j-1)/2] - i_m$ $(i_m \geqslant 1)$, and that at least $m - i_m$ of these elements are odd.

Repeating the preceding argument for $C_{m-1}, C_{m-2},...$, each time we reach the point where we can assume that $|C_n \cap S| = [(j-1)/2] - i_n$ $(i_n \geqslant 1)$ and that at least $m - \sum_1^n i_l$ of these elements are odd. If $m - \sum_1^n i_l > 0$ we go to C_{n-1}, otherwise we stop and assume that for $1 \leqslant l < n$, $|C_l \cap S| = [(j-1)/2] - i_l$ and $i_l = 0$. In any case, $m - \sum_1^m i_l \leqslant 0$ and, since $|C_0 \cap S| = (k/2) - [m(j-1)/2]$, we have

$$S(k-1) = \sum_0^m |C_l \cap S|$$

$$= \frac{k - m(j-1)}{2} + \sum_1^m \left(\frac{j-1}{2} - i_l\right) + m \leqslant k/2. \quad \blacksquare$$

The above two theorems provide expressions for $\mu(M)$ when k is larger that a certain minimum value. If k does not satisfy this requirement, the conclusions of the theorems will still be valid if we replace equalities with inequalities whose direction depends on the parity of j. If j is even, $\mu(M)$ is no larger than the value given in Theorem 3, since the proof that $\mu(M) \leqslant d$ is still valid. If, on the other hand, j is odd, $\mu(M)$ is at least as large as the value given in Theorem 4, since the construction of the M-sequence is still valid.

The sequences constructed in Theorem 4, where j is odd and k even, are not, of course, unique. The following remark shows that in this case there exists a large number of nonequivalent (that is, with periods that do not differ by a cyclic permutation) M-sequences of density $\mu(M)$.

Remark 3. If $M = \{1, j, k\}$, where $j > 1$ is odd and $k \geqslant j(j-1)/2$ is even, then there exist at least

$$2^{[(j-1)/2]-1}$$

nonequivalent M-sequences of density $\mu(M)$ and period length $k + j$.

Proof. The case $j = 3$ being trivial, we shall be concerned with the case $j \geqslant 5$. We consider a periodic binary sequence with period

$$(10)^{(k/2)-e}\,0A_1A_2 \cdots A_{e-2}010^{j-2e+1}10B_1B_2 \cdots B_{e-2}00,$$

where $e \in \{2, 3,..., (j-1)/2\}$ and for each i, either $A_i = 00$ and $B_i = 10$ or $A_i = 10$ and $B_i = 00$. It is a simple matter to verify that the resulting sequence is an M-sequence of density $\mu(M)$, and that there are $1 + 2 + 2^2 + \cdots + 2^{[(j-1)/2]-2}$ such sequences. The stated number is obtained if to these we add the sequence constructed in Theorem 4. ∎

The following remark shows the existence of an M-set of density $\mu(M)$ and surprisingly short period in a special case of Theorem 4.

Remark 4. If $M = \{1, j, k\}$ where $j = 4n + 1$ and $k = j(j-1)/2$ then a periodic sequence with period $(10)^n\,0$ is an M-sequence of density $\mu(M)$ and period length $(j + 1)/2$.

Proof. We observe that $k \equiv 1 \pmod{2n+1}$ and $j \equiv -1 \pmod{2n+1}$ and that

$$\delta(S) = n/(2n+1) = (k/2)/(k+j) = \mu(M). \quad ∎$$

We conclude this section with two theorems giving $\mu(M)$ in the special cases in which $k = j + 1$ or $k = j + 2$.

THEOREM 5. *If* $M = \{1, n, n+1\}$ *then*

$$\begin{aligned} \mu(M) &= (2n/3)/(2n+1) && \text{if } n \equiv 0 \pmod 3,\\ &= \tfrac{1}{3} && \text{if } n \equiv 1 \pmod 3,\\ &= [(n+1)/3]/(n+2) && \text{if } n \equiv 2 \pmod 3. \end{aligned}$$

Proof. (i) If $n \equiv 0 \pmod 3$, then a periodic sequence with period $10(100)^{n/3-1}\,10000(100)^{n/3-1}$ is an M-sequence of the stated density. Conversely, if S is an M-set with $0 \in S$, let $e = S(n-1)$. If $e \leqslant n/3$, then

$$S(n+1) = S(n-1) = e \leqslant (n+2)(2n/3)/(2n+1).$$

If $e \geqslant (n/3) + 1$, then $\{n+2,..., 2n\}$ can contain at most $n - 2e + 1$ elements of S, hence $S(2n) \leqslant n - e + 1 \leqslant 2n/3$. In either case the result follows from Lemma 1.

(ii) If $n \equiv 1 \pmod 3$, the set consisting of multiples of 3 is an M-set of density $\frac{1}{3}$. Conversely, $\frac{1}{3}$ is an upperbound for $\mu(\{i, j, i+j\})$, as mentioned in the discussion following Remark 2.

(iii) If $n \equiv 2 \pmod 3$, then a periodic sequence with period $(100)^{(n+1)/3}0$

is an M-sequence of the stated density. Conversely if S is as before, let $S(n+1) = [(n+1)/3] + e$. If $e \leqslant 0$, the result is immediate. Otherwise $\{n+2,..., 2n\}$ cannot contain more than $[(n+1)/3] - 2e$ elements of S, and therefore

$$S(2n) \leqslant \frac{2(n+1)}{3} - e \leqslant \frac{2n-1}{3} \leqslant (2n+1)\frac{(n+1)/3}{n+2}. \quad \blacksquare$$

THEOREM 6. *If* $M = \{1, n, n+2\}$ *then*

$$\begin{aligned} \mu(M) &= \tfrac{1}{2} && \text{if } n \text{ is odd,} \\ &= (n/2)/(n+1) && \text{if } n \text{ is even.} \end{aligned}$$

Proof. The case where n is odd does not require any comment. If n is even, then a $\{1, n\}$-set of period $n+1$ is also an M-set, since $n+2 \equiv 1 \pmod{n+1}$. Therefore

$$\mu(M) = \mu(\{1, n\}) = (n/2)/(n+1). \quad \blacksquare$$

4. THE FAMILY $M = \{1, 2, j, k\}$

We begin the consideration of the family $M = \{1, 2, j, k\}$ with a theorem that gives $\mu(M)$ when j is not a multiple of 3.

THEOREM 7. *If* $M = \{1, 2, j, k\}$, *where* $2 < j < k$ *and* $j \not\equiv 0 \pmod 3$, *then*

$$\begin{aligned} \mu(M) &= \tfrac{1}{3} && \text{if } k \not\equiv 0 \pmod 3, \\ &= (k/3)/(k+2) && \text{if } k \equiv 0 \pmod 3 \text{ and } j \equiv 1 \pmod 3, \\ &= (k/3)/(k+1) && \text{if } k \equiv 0 \pmod 3 \text{ and } j \equiv 2 \pmod 3. \end{aligned}$$

Proof. (i) When $k \not\equiv 0 \pmod 3$, the set $\{0, 3, 6,...\}$ is an M set of density $\frac{1}{3}$, which cannot be exceeded since $\mu(\{1, 2\}) = \frac{1}{3}$.

(ii) If $k \equiv 0 \pmod 3$ and $j \equiv 1 \pmod 3$, let the period of a periodic sequence be $(100)^{k/3}00$. Then this will be an M-sequence of the desired density. Conversely, let S be an M-set with $0 \in S$ and such that $S(k+1) \geqslant (k/3) + 1$. We decompose $\{0, 1,..., k+1\}$ into sets $A_i = \{3i, 3i+1, 3i+2\}$ and $B = \{k, k+1\}$. It is then clear that $|A_i \cap S| \leqslant 1$ and $k \notin S$, so that we necessarily have $k+1 \in S$. This implies $k-1 \notin S$ and consequently $3i+2 \notin S$ for all $i < k$. Let i_0 be the smallest i for which $3i+1 \in S$. Then $3i \in S$ for $i < i_0$ and $3i+1 \in S$ for $i \geqslant i_0$. We note that $k+1 \in S$ implies $k-j+1 \notin S$, and since $k-j+1 \equiv 0 \pmod 3$ such an i_0 must exist and in fact satisfy $3i_0 \leqslant k-j+1$. Then $3i_0 + j - 3 \in S$ and $3i_0 + j - 3 \leqslant$

$k - 2$, hence $3i_0 - 3 \notin S$ contrary to the choice of i_0. From this contradiction we obtain $S(k + 1) \leqslant k/3$, hence the desired result.

(iii) For the remaining case, let the period of a periodic sequence be $(100)^{k/3}\, 0$. Then this will be an M-sequence of the stated density. Conversely, a decomposition of $\{0, 1, \ldots, k\}$ into A_i as in (ii) and $\{k\}$ implies that $S(k) \leqslant k/3$. ∎

The behavior of $\mu(M)$ when j is a multiple of 3 is not completely known. If $\bar{k}$ is the residue of k modulo $j + 1$, the following theorem describes $\mu(M)$ except in the case $\bar{k} \equiv 1 \pmod 3$, which is discussed in the succeeding remarks.

THEOREM 8. *If* $M = \{1, 2, j, k\}$, $2 < j < k$, *and* $j \equiv 0 \pmod 3$, *then*

$$\begin{aligned} \mu(M) &= (j/3)/(j + 1) \qquad \text{if} \quad \bar{k} = 1 \text{ or } j \text{ or } \bar{k} \equiv 2 \pmod 3, \\ &= \{[k/(j + 1)] \cdot (j/3) + (\bar{k}/3)\}/(k + 1) \qquad \text{if } \bar{k} \equiv 0 \pmod 3, \quad \bar{k} \neq j. \end{aligned}$$

Proof. (i) When $\bar{k} = 1$ or j or $\bar{k} \equiv 2 \pmod 3$, the $\{1, 2, j\}$-sequence of Theorem 3 is also an M-sequence of the desired density.

(ii) If $\bar{k} \equiv 0 \pmod 3$ and $\bar{k} \neq j$, a periodic sequence with period $((100)^{j/3}0)^n(100)^{\bar{k}/3}0$, where $n = [k/(j + 1)]$, is an M-sequence, a fact that can be easily verified. The converse follows from the decomposition $\{0, 1, \ldots, k\} = \bigcup_{i=0}^{n-1} A_i \cup \bigcup_{i=0}^{\bar{k}/3} B_i \cup \{k\}$, where the A_i are consecutive sets of $j + 1$ consecutive integers and the B_i consecutive sets of three consecutive integers, if we observe that $|A_i \cap S| \leqslant j/3$, $|B_i \cap S| \leqslant 1$, and $k \notin S$ if $0 \in S$. ∎

Remark 5. If $\bar{k} \equiv 1 \pmod 3$ and $\bar{k} \geqslant 7$, computed examples indicate that

$$\mu(M) = \{(nj/3) + [\bar{k}/3]\}/(k + 1),$$

where $n = [k/(j + 1)]$, achieved by sequences with period

$$((100)^{j/3}0)^n(100)^{k/3}\, 00.$$

Unfortunately, the methods used in previous results cannot be used here to show that this value cannot be exceeded, since there are M-sets S with period length of the order of $4k$ and such that $S(k) = (nj/3) + [\bar{k}/3] + 1$, but with density not exceeding the conjectured $\mu(M)$. The proof of this, therefore, remains an unsolved problem.

The case in which $\bar{k} = 4$ contains the first known family for which not only $\mu(M) > d(M)$, but also the denominator of $\mu(M)$ in lowest terms does not divide the sum of two elements of M:

Remark 6. If $M = \{1, 2, 3n, 3n + 5\}$ $(n \geqslant 2)$, then

$$\mu(M) = (2n + 1)/(6n + 8).$$

Proof. If S is a periodic sequence with period $((100)^n 0)^2\, 010000$, then it can be verified that S is an M-set of the stated density. For the converse, suppose S is an M-set with $0 \in S$ and $S(6n + 7) \geqslant 2n + 2$. If we write $\{0, 1,..., 6n + 7\} = A_1 \cup A_2 \cup B$, where $A_1 = \{0, 1,..., 3n\}$, $A_2 = A_1 + 3n + 1$ and $B = \{6n + 2,..., 6n + 7\}$, the proof of Theorem 3 guarantees that $| A_i \cap S | \leqslant n$, so we must have $| B \cap S | = 2$. The proof is completed by deriving a contradiction for every choice of two elements for $B \cap S$. This verification is straightforward but lengthy, and is therefore omitted in this presentation. ∎

We conclude with another example of a family of sets for which $\mu(M) > d(M)$:

Remark 7. If $M = \{1, 3, 4, k\}$ and $k \equiv 2 \pmod 7$, then

$$\mu(M) = 2[k/7]/(k - 1).$$

Proof. A sequence with period $(1010000)^{[k/7]}\, 0$ proves that $\mu(M)$ is at least as large as claimed. For the reverse inequality write $\{0, 1,..., k - 2\} = \bigcup_{i=0}^{[k/7]-1} A_i \cup \{k - 2\}$, where $A_i = \{0, 1,..., 6\} + 7i$ and let S be an M-set with $0 \in S$; it is clear that $| A_i \cap S | < 2$. If $k - 2 \notin S$, then $S(k - 2) \leqslant 2[k/7]$. If $k - 2 \in S$, then, $k, k - 1, k + 1, k + 2 \notin S$, so

$$S(k + 2) \leqslant 2[k/7] + 1 \leqslant (k + 3) \cdot 2[k/7]/(k - 1).$$

In either case, the result follows from Lemma 1. ∎

References

1. D. G. Cantor and B. Gordon, Sequences of integers with missing differences, *J. Combinatorial Theory* (*A*) **14** (1973), 281–287.
2. T. S. Motzkin, unpublished problem collection.